The Smarties-Box Challenge:

Supporting systematic approaches to problem solving

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The four proficiency strands are employed in this mathematically rich estimation task which combines estimation skills and multiplicative thinking within a challenging problem solving context.

Can your students work out how many Smarties there might be in the Smarties-box? The Smarties-Box Challenge encourages students to apply several different mathematical capabilities and concepts—such as, estimation, multiplication, and the notion of being systematic—to solve a complex, multistep problem.

Overview of the mathematics

Reconsidering estimation: Striving for precision under uncertainty

Enhancing estimation skills through repeatedly and explicitly exposing students to problems where an exact answer is either unnecessary or too complex to rapidly calculate is critical to developing number sense (Reys et al., 2012). However, the concept of estimation is also clearly relevant in a problem-solving context where there is insufficient information to perform a precise calculation. Indeed, such a conceptualisation of estimation is central to a diverse range of applied fields which draw heavily on mathematics, from traffic management (e.g., Zhao, Ge, Wang & Zu, 2006) to economics (Green & Porter, 1984).

The Smarties-Box Challenge is a mathematical task designed to get primary school age children to think about making decisions under uncertainty through requiring them to combine several different mathematics skills and capabilities to estimate how many Smarties are in a tissue box (known as the Smarties-box). To effectively engage in the Smarties-Box Challenge, students are required to demonstrate aspects of all four proficiency strands identified in the *Australian Curriculum: Mathematics*—problem solving, understanding, reasoning and fluency; implied criteria for a "mathematically rich investigative task" (Day & Hurrell, 2013, p.4).

Beyond estimating, there are several additional mathematical concepts embedded in the task including measurement concepts and the notion of an average when working with data. However, at its core, the Smarties-Box Challenge attempts to combine estimation skills with multiplicative thinking and systematic problem solving.

What is multiplicative thinking?

Multiplicative thinking involves transitioning away from a 'groups of' model (which characterises additive thinking) and moving towards a 'factor–factor–product' model (Siemon, 2013). Day and Hurrell (2015) argue that teaching with arrays can help support this transition. The authors explain how arrays can be used to illustrate both the commutative property (e.g., 4×3 is revealed to be equivalent to 3×4 through rotating the array) and the distributive property (e.g., $6 \times 4 = 5 \times 4 + 1 \times 4$) of multiplication. Moreover, they describe how arrays, through offering an area model of multiplication, support the transition from more concrete additive thinking to more abstract multiplicative thinking.

Other authors have shown how the emergence of multiplicative thinking can also be supported through engaging students in problem-solving type activities, and encouraging them to reflect on their reasoning. For example, Empson and Turner (2006) demonstrated how the process of interviewing students about their approach to a paper-folding investigation appeared to shift student's thinking away from the additive and towards the multiplicative, through enabling students to make connections between their actions and subsequent outcomes. It is suggested that the Smarties-Box Challenge may support the development of multiplicative thinking through a similar process; that is, through

encouraging students to reflect on their reasoning when engaged with a problem-solving activity that has a multiplicative structure.

As will become apparent later in the article when describing some of the various student approaches, although it is possible to comprehensively engage with the Smarties-Box Challenge without thinking multiplicatively (see Caleb's group), relying on repeated addition is revealed to be less efficient and less likely to result in reliable estimates. Moreover, the structure of the Smarties-Box Challenge rewards possessing a comprehensive working knowledge of the distributive property (see Will's group), which has been described as being at the core of multiplication (Kinzer & Stafford, 2014).

What is systematic problem solving?

George Polya (1945) introduced four basic principles of problem solving: understand the problem, devise a plan, carry out the plan and look back. Keeping these principles in mind when approaching a problem-solving task, such as the Smarties-Box Challenge, encourages students to be systematic and methodical in their thinking. References will be made to each of these four problem solving principles during the description of how our Year 3/4 class went about solving the Smarties-Box Challenge.

The Smarties-Box Challenge

Figure 1. The Smarties-Box Challenge.

How many individual Smarties are there in this box? How would you begin to work it out?

Setting up the challenge

Materials

- 7 bags of fun-size Smarties packets (which provides 77 packets)
- An empty tissue box
- Paper and pencils
- Counters and hundreds charts
- Rulers or measuring tapes
- Digital scales (if available)

Pre-planning: Creating the Smarties-box

Two rows of seven fun size packets of Smarties fit snugly into a tissue box. Although four of these 2-by-7 layers of packets will exactly fill the box to the top, you may want to provide room for students to peer inside the Smarties-box.

Indeed, to allow students to peek into the box, the Smarties-box constructed for the investigation described in the current article had three layers of Smarties packets, except for each end of the Smarties-box, which had four layers (see Figure 1). This meant that there were 46 fun-size packets in the Smarties-box altogether (14 packets per layer × 3 layers = 42 packets; plus four additional packets, two at each end).

It is recommended that you have at least one fun-sized packet of Smarties per student left over (these packets will be needed when the challenge begins).

The Smarties-Box Challenge is made more complex (and interesting) by the fact that there is not a uniform number of Smarties in each fun-size packet. In fact, this variability is at the heart of the Smarties-Box Challenge. For the current investigation, some packets contained as few as 10 Smarties, whilst other packets had as many as 14 Smarties, with the average being just over 12 Smarties per packet. There were 558 Smarties in the 46 packets in total.

It is at this point that you need to make a decision. Obtaining the above knowledge about the exact number of Smarties in the 46 packets was obviously quite time consuming. It involved counting the individual Smarties in each packet and resealing each packet with sticky-tape, before placing them back into the tissue box! Consequently, if you wish to use 558 as your total (and you can be 95% confident that the actual total number of Smarties in 46 packets chosen at random is between 547 and 569) and save yourself the trouble of counting them, feel free.

Rules for the Smarties-Box Challenge

Students were provided with the following set of rules when they undertook the challenge:

1. Students work in groups of four.
2. Each student starts with one fun-size packet of Smarties and can use any other mathematical equipment (except calculators!) they anticipate might be helpful.
3. Groups have 25 minutes to work on their estimates after which time they must submit their estimate.
4. Only one group member can look at the Smarties-box at a time. You may touch the outside of the Smarties-box (e.g., to measure it), but not the inside. You can weigh the Smarties-box if you wish.
5. You and your group members may eat your Smarties at any time (although you might also use them to help you with the challenge!).

Teachers should take some time to discuss these rules as a class, and ensure that students are clear about what the task requires. This can be linked to Polya's (1945) principle of students understanding the problem. Some potentially useful questions to pose to students to support them in making sense of the problem include:

- What are you being asked to do? Can you explain the task to me in your own words?
- Do you understand the language used in the problem and in the rules? Are there any words that you find confusing?
- What mathematics do you think you might need to use when working on the challenge?
- What materials and resources might be particularly helpful?

At this stage, students are ready to begin working in their groups.

Supporting student thinking

Prompts for getting started

Depending on the grade level and mathematical confidence of the students you are working with, as they are progressing on the problem, you may wish to provide them with some prompting questions to encourage them to approach the problem systematically. These prompts have been developed to support students as they transition into Polya's (1945) second principle of problem solving: devise a plan.

It is important to note that these prompting questions are best used relatively sparingly, and are particularly valuable for students who are struggling to make meaningful progress in even beginning the challenge. Many groups will develop their own idiosyncratic approach for solving the task. This diversity of approaches should be embraced while students are working on the problem, with the subsequent mathematical discussion providing opportunities for student approaches to be compared, contrasted and evaluated. Prompting questions:

- How many Smarties-packets do you think there are in the box?
  - Further prompts: How many Smarties-packets are in each layer? How many layers do you think there are? Could you use the Smarties-box or a ruler to help you work out how many layers there are? Do all parts of the box have the same number of layers?
- How many Smarties do you think there are in each packet?
  - Open your Smarties packets and count your Smarties. Do all group members have the same number of Smarties in their packets (highly unlikely)? Maybe you should check with some other groups about how many Smarties are in each packet? On average, how many Smarties do you think there are in a packet?
- How many Smarties do you think there are in total?
  - If you think you know how many packets of Smarties there are, and about how many Smarties are in each packet, how could you use this information to work out how many Smarties there are in total?

Solving the Smarties-Box Challenge

When we tackled the Smarties-Box Challenge in our classroom (Year 3/4), groups tended to adopt very different approaches for trying to develop an accurate estimate. Some of their approaches are described below. It is clear from reading the description of how students went about the task that groups tended to cycle between Polya's (1945) principles of devising a plan, and carrying out the plan. Essentially students discarded or revised aspects of their informal problem solving plans when they encountered obstacles. Such flexibility is how 'real mathematics' is done, and should be encouraged by the teacher.

Caleb's group

After interrogating the Smarties-box and counting packets individually, Caleb's group decided that there were "around 50 packets" of Smarties in the box, and "around 12 Smarties" in each packet. They conceptualised the problem as involving multiplication, however did not know initially how to calculate 50

groups of 12. One group member decided to skip-count by 12s on a 120s chart, although abandoned this path once they moved past 120, noting “that’s only 10 packets of Smarties and I’ve already run out of room on my chart!” Another group member had the idea of drawing 50 groups with 12 dots in each. However, once he started drawing the problem, he realised that “we only have to make 25 groups, and then we can double it”. Appreciating that drawing the problem this way seemed far more manageable than using a 120s chart, the whole group got thoroughly behind this approach.

After drawing the 25 groups of 12, the group members then counted the dots by ones, erasing them as they counted (see Figure 2). In total, they counted 292 dots, which, following a rigorous discussion, they eventually accurately doubled, using place-value partitioning (that is, 200 + 200, 90 + 90, 2 + 2). The final estimate put forward by Caleb’s group was that there were 584 Smarties in total. This was an impressively accurate estimate given that the group had tackled the problem without employing multiplication.

Figure 2. Caleb’s group represented the Smarties-Box Challenge pictorially.

Sam’s group

Sam’s group worked out how many Smarties packets there were in the box by first counting 14 packets on each layer (“there are 7 packets per row, and there are 2 rows”). Next, after stacking up the Smarties packets next to the Smarties-box, they decided that there were “probably” three layers of packets. The group computed this to be 42 packets in the box in total (“14 plus 14 is 28; add another 14 is 42”).

After opening their four packets of Smarties and finding 11, 11, 12 and 13 Smarties, they decided that “usually” there were 11 Smarties in each packet, and used this as their average. They conceptualised the problem as 42 groups of 11. None of the group members was confident in how this computational problem could be worked out mentally (and none of them suggested using pencil and paper), so Sam suggested they use the unifix and teddies to help with the problem. This suggestion was met with excitement. Another one of the group members recalled the commutative property of multiplication (“42 groups of 11 is the same as 11 groups of 42”), and decided that “making 11 groups of 42 is easier, so let’s do it that way”.

The group then compiled 11 groups of 42 and added them altogether (see Figure 3), organising the unifix and teddies into lots of 10 (and then the lots of 10 into lots of 100) as they calculated. In total, they counted “around 460”, and were about to put this forward as their estimate when Sam noticed the four extra Smarties packets comprising the fourth layer. He yelled “we have to add 44, because we forgot about those boxes”. As the 25 minutes was elapsing, the group quickly changed their estimate to 504, feeling very confident that they had got the ‘right’ answer.

Figure 3. Sam’s group work on the Smarties-Box Challenge using unifix and teddies.

Will’s group

Will’s group, which was full of highly capable mathematical thinkers, were surprisingly rather casual in how they initially approached the challenge. All group members agreed from looking at the Smarties-box that there could “only be three layers of packets… plus the extra packets on the top”, and they sent one of their group members over to work out how many boxes there were in total. After rapidly counting by 2s, this group member confidently declared that there were 48 packets altogether.

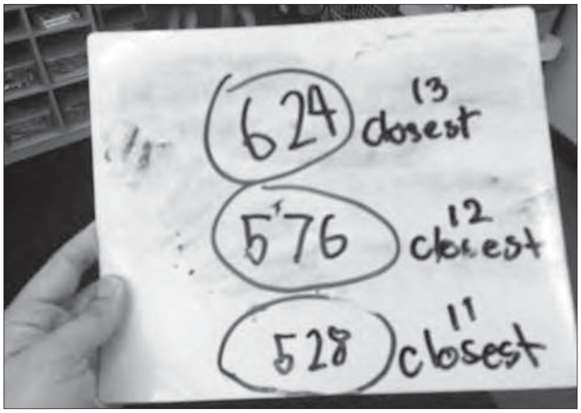

Figure 4. Will's group calculate three different estimates.

Will's group were highly methodical in their approach to the remainder of the problem. After opening their Smarties packets and finding 11, 12, 12, and 13 Smarties in each, they decided to calculate three different estimates, one for each potential 'average' (see Figure 4). They appropriately applied the distributive property of multiplication to calculate 48 by 11 ("48 by 10 equals 480, plus another 48 equals 528"), and then added 48 more to work out the total number of Smarties if 12 were the average (576), and another 48 to work out the total number if 13 were the average (624). The group then had the inspired idea to survey the rest of the class to work out what the 'best' average was for the number of Smarties in a packet. After talking to another three groups, they concluded that 12 was the best average, but the real average "was probably a little more than 12". As a result, they rounded up their estimate to 580. Although the accuracy of their final estimate was somewhat undermined by their casual approach to calculating the number of Smarties packets in the Smarties-box, the quality of the mathematical thinking displayed by Will's group on the second half of the problem was impressive.

Post-challenge discussion

After students submitted their estimates, we came together to discuss our various approaches to the challenge. This resonated with Polya's (1945) fourth principle of problem solving: looking back. This stage involves the teacher facilitating a reflective discussion, as students explore "what worked and what didn't" with their various approaches.

Some key guiding questions to encourage groups to reflect on their estimates through a critical lens included:

- How did you go about solving the challenge?
- How confident are you in your estimate?
- What parts of your estimate are you certain about and what parts are you uncertain about?
- What additional information would you need to improve your estimate?
- How could you get this information?
- Now you have listened to other groups discuss how they went about the challenge, are there any things you would change if you were to undertake a similar challenge again? Was there a more efficient way of approaching the problem?

Obviously in our classroom, Will's group had effectively worked through most of these questions themselves in their group without prompting. Their decision to systematically establish a range of possible estimates, and then survey other class members to obtain additional information and reduce the level of uncertainty in their estimate, was inspired. However, their rather casual estimate of the number of packets in the Smarties-box allowed other students, such as those in Sam's group, to have input into how this first component of the problem could also be approached more systematically (see Figure 5).

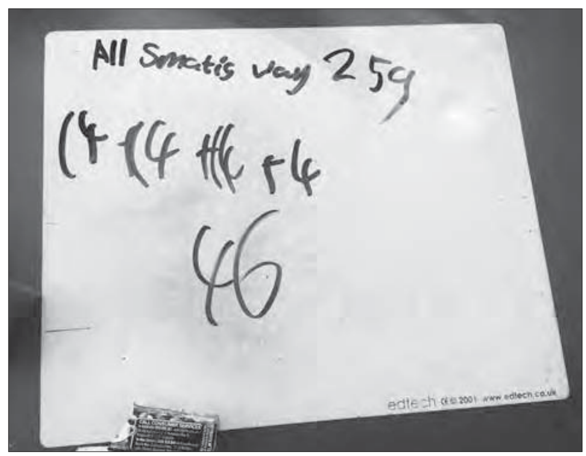

Figure 5. Sam's group shows Will's group how to approach the first part of the challenge more systematically.

At the end of the investigation, Cameron's group had obtained the closest estimate, accurately working out that there were 46 packets of Smarties in the box, and around 12 Smarties per packet, and then multiplying out the answer using the distributive property ("46 by 10 is 460; and double 46 is 92. Together it equals 552"). Their estimate of 552 was, in fact, only 6 Smarties off the actual answer. Cameron and his peers received the Smarties-box itself as a prize, although thankfully they chose to share it with the rest of the class as well!

Conclusion

Good problem-solving tasks require students to integrate multiple mathematical proficiencies (Day & Hurrell, 2013). The Smarties-Box Challenge requires students to employ a diverse range of mathematical skills and capabilities, as they work together as a team to solve quite a complex estimation problem. A brief summary of some of the ways the task connects to the four proficiency strands in the *Australian Curriculum: Mathematics* is provided below:

- **Problem Solving:** Students need to consider when they can know something for 'certain' and calculate with precision, and when imperfect information means that calculating with precision is an impossibility. Moreover, students explore how they can reduce uncertainty through pooling data and increasing the amount of information they have at their disposal.
- **Understanding:** Students need to conceptualise the task as one involving multiplication. The task rewards the capacity to think multiplicatively, and the post-task discussion may provide a context for shifting students away from additive thinking and towards the beginning stages of multiplicative thinking (Day & Hurrell, 2015). For example, the teacher could draw attention to the commutative property of multiplication in contrasting the approach adopted by Caleb's group and Sam's group. Whilst the former were still predominantly entrenched in the additive approach, unable to reconceptualise 50 groups of 12 as 12 groups of 50 (despite knowing that their chosen approach was inefficient), the latter were able to reconstruct 42 groups of 11 as 11 groups of 42. There would then be a further opportunity to contrast the more elementary multiplicative thinking involved in the approach of Sam's group with the more sophisticated thinking demonstrated in Will's group, who expertly employed the distributive property to place themselves in the position to arrive at the best possible estimate.
- **Reasoning:** Working in groups promotes constructive discussion and encourages students to justify their decisions mathematically. Reasoning is also promoted through reflecting on the problem, and the problem-solving process, during the whole-class discussion (Day & Hurrell, 2013).
- **Fluency:** The problem encourages students to apply the distributive property of multiplication, and students with fluent knowledge of their multiplication facts are more likely to be both more efficient and accurate with their final estimate.

As a final word, undertaking the Smarties-Box Challenge is also great fun and tends to engage the whole-class (not least because of the Smarties 'dangling' as a prize at the end). It is a highly worthwhile activity for primary school students in Year 3 onwards.

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